

PERIODIC HOMEOMORPHISMS OF 3-MANIFOLDS FIBERED OVER S^1

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ABSTRACT. Two problems concerning periodic homeomorphisms of 3-manifolds are considered. The first is that of obtaining systems of incompressible surfaces invariant under a given involution. The second problem is the realization by a periodic homeomorphism of an element of finite order in the mapping class group of a 3-manifold. Solutions to both problems are obtained in certain instances.

1. Introduction. In this paper we are concerned with (1) the problem of finding systems of incompressible surfaces invariant under a given involution of a 3-manifold and (2) the construction of periodic maps within a given homotopy class of finite order. For a class of closed 3-manifolds fibered over S^1 we prove that if h is a map such that h^p is homotopic to the identity map (for p prime) then there exists a PL map g homotopic to h such that g^p is the identity map. We also obtain an explicit characterization of all PL involutions on these 3-manifolds.

In studying involutions of 3-manifolds it is usually helpful if one can equivariantly split the 3-manifold into simpler pieces. For sufficiently large 3-manifolds (those which contain two-sided incompressible surfaces) one would be apt to attempt such a splitting along incompressible surfaces. Our first theorem enables us to perform this kind of equivariant reduction in certain circumstances. For additional results concerning this type of equivariant splitting for involutions the reader may consult [5], [7], and [12].

Recall that a 3-manifold M is P^2 -irreducible if every polyhedral 2-sphere in M bounds a 3-cell and M contains no two-sided projective planes. A closed surface F (not equal to a 2-sphere) is *incompressible* in M provided that whenever there exists a disk D such that $D \cap F = \partial D$, then it follows that ∂D also bounds a disk in F .

THEOREM 1. *Let M be a compact P^2 -irreducible 3-manifold and let h be a PL involution of M . Suppose that F is a two-sided, closed, orientable, incompressible surface in M . Then there exists a PL map g homotopic to h such that g^2 is the identity map on F .*

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pressible surface in M such that $h(F)$ is homotopic to F in M . Then there exists an isotopy of M carrying F to a surface G such that G is transverse to $\text{Fix}(h)$ and either $h(G) \cap G = \emptyset$ or $h(G) = G$.

This theorem can be readily applied to the class of 3-manifolds which we now describe. Given a homeomorphism ϕ of a closed surface F , we let $M(\phi) = F \times R^1 / \phi$ denote the 3-manifold obtained from $F \times R^1$ by identifying (x, t) with $(\phi(x), t + 1)$. We designate the points of $M(\phi)$ by $[x, t]$. There is the fibering of $M(\phi)$ over S^1 defined by $[x, t] \rightarrow [t]$, where we view the circle S^1 as R^1/Z . The fiber over the point $[t]$, denoted by $[F \times t]$, is an incompressible surface in $M(\phi)$ [13]. We are interested in those 3-manifolds $M(\phi)$ for which $H_1(M(\phi); Q) \cong Q$. This is equivalent to requiring $\det(I - \phi_*) \neq 0$, where ϕ_* is the automorphism of $H_1(F; Q)$ induced by ϕ . If $h: M(\phi) \rightarrow M(\phi)$ is a homeomorphism, we show (§3) that there exists a homotopy carrying $h([F \times 0])$ back to $[F \times 0]$.

THEOREM 2. *Let h be a PL involution on $M(\phi) = F \times R^1 / \phi$, where F is a closed orientable surface and $H_1(M(\phi); Q) \cong Q$. Then h is equivalent to an involution h' defined on $M(\Psi) = F \times R^1 / \Psi$ by $h'([x, t]) = [\beta(x), \lambda(t)]$, where β is some homeomorphism of F , the map Ψ is isotopic to ϕ , and $\lambda(t) = t, 1 - t$, or $t + \frac{1}{2}$.*

To prove this theorem we apply Theorem 1 to find an invariant system of incompressible surfaces along which we split $M(\phi)$ to obtain copies of $F \times I$ (we let I denote the unit interval $[0, 1]$). Then we apply the following result of P. Kim and Tollefson to the involution induced by h on these simpler pieces.

THEOREM 3 [5]. *Let F be a compact surface and let h be a PL involution of $F \times I$ such that $h(F \times \partial I) = F \times \partial I$. Then there exists a map β of F (with $\beta^2 = 1$) such that h is equivalent to the involution h' of $F \times I$ defined by $h'(x, t) = (\beta(x), \lambda(t))$, for $(x, t) \in F \times I$ and $\lambda(t) = t$ or $1 - t$.*

Now let us turn to the problem of constructing periodic maps within a given homotopy class of finite order. J. Nielsen in [8] proved that each mapping class (of orientation preserving homeomorphisms of a compact orientable surface) of finite order contains a periodic homeomorphism. This is extended in [14] to arbitrary maps of any closed surface. One might have expected a similar situation for 3-manifolds. Recently, however, F. Raymond and L. Scott [9] constructed a 3-manifold for which this nice behavior fails. Their example is a torus bundle over S^1 . In view of our Theorem 5 it appears that among closed orientable 3-manifolds fibered over S^1 this failure of Nielsen's theorem is probably restricted to those with torus fibers.

The question we are interested in can be posed in the following way. Let $\text{Out}(\pi_1(M))$ denote the group of outer automorphisms of the closed, connected,

aspherical 3-manifold M (that is, the quotient group of the automorphism group of $\pi_1(M)$ by the subgroup of inner automorphisms). There is the natural homomorphism $\Psi: \text{Homeo}(M) \rightarrow \text{Out}(\pi_1(M))$. Given a finite subgroup H of $\text{Out}(\pi_1(M))$, when does $\Psi^{-1}(H)$ contain a finite group of homeomorphisms G such that $\Psi|_G: G \cong H$? That is, which of the finite subgroups of $\text{Out}(\pi_1(M))$ are actually realized as groups of homeomorphisms of M ? If M is a closed aspherical manifold and $\pi_1(M)$ is centerless, then by a theorem of A. Borel [4], the restriction of Ψ to any finite group of homeomorphisms is a monomorphism. Thus, in this situation, all finite groups of homeomorphisms are faithfully represented in $\text{Out}(\pi_1(M))$.

We obtain the following result in this direction for the special case when H is a finite cyclic group.

THEOREM 4. *Suppose that $M(\phi) = F \times R^1/\phi$, where F is a closed surface of negative Euler characteristic and $H_1(M(\phi); \mathbb{Q}) \cong \mathbb{Q}$. If $h: M(\phi) \rightarrow M(\phi)$ is a map such that $h^p \simeq 1$ (for some prime p) then there exists a PL homeomorphism g of $M(\phi)$ such that $g \simeq h$ and $g^p = 1$.*

From this it follows that a periodic homeomorphism of $M(\phi)$ of prime period (perhaps with a wildly embedded fixed point set) is homotopic to a periodic PL map. The question as to when an involution of a 3-manifold is PL has recently been settled by K. Kwun in [6] who proves that any involution of a closed 3-manifold with a tame fixed point set is already PL.

2. Proof of Theorem 1. In this section everything is done in the PL category. Given an involution h of a 3-manifold M and an incompressible surface F which is homotopic to $h(F)$ in M , we want to improve the embedding of F with respect to h . More precisely, we want to move F by an isotopy of M such that afterwards we have either $h(F) \cap F = \emptyset$ or $h(F) = F$. The surface F is *homotopic* to $h(F)$ in M if there is a map $G: F \times I \rightarrow M$ such that $G(F \times 0) = F$ and $G(F \times 1) = h(F)$. We say that F is *isotopic* to $h(F)$ in M if there exists a map $G: M \times I \rightarrow M$ such that for $G_t = G|_{M \times t}$ we have $G_t: M \rightarrow M$ a homeomorphism, $G_0 = \text{identity}$, and $G_1(F) = h(F)$.

Consider a simplicial involution h on a triangulated 3-manifold M . Let F be a closed surface embedded in the interior of M as a subcomplex. We move the surface F into h -general position [12] by the following procedure. First, move F into general position with $\text{Fix}(h)$, the fixed-point set of h . Then move $F - \text{Fix}(h)$ into general position with $h(F) - \text{Fix}(h)$ by an isotopy of M that is constant on $\text{Fix}(h)$. Afterwards $F \cap h(F)$ will be a graph whose branch points, which we shall refer to as *special vertices*, are contained in $F \cap \text{Fix}(h)$. We define the complexity $c(F)$ of F to be $a + b$, where a is the number of components in

$F \cap h(F) - \text{Fix}(h)$ and b is the number of components in $F \cap \text{Fix}(h)$.

We use techniques similar to those introduced in [12]. The idea is that if we have a surface in h -general position whose complexity is greater than 0, then in given situations we can isotope it to one that is either invariant under h or of a lower complexity. We have restricted our attention to orientable surfaces mostly as a matter of convenience for the proof.

PROOF OF THEOREM 1. Let Σ denote the set of all surfaces in M which are isotopic to F and in h -general position. The set Σ is not vacuous since F itself may be moved into h -general position. We choose a surface S in Σ that has minimal complexity. If $c(S) = 0$, then $h(S) \cap S = \emptyset$ and there is nothing to show. However if $c(S) > 0$, we shall show that this implies the existence of an invariant surface in M isotopic to F . So let us assume that $h(S) \cap S \neq \emptyset$.

The surface S fails to be in general position with $h(S)$ at the special vertices of $S \cap h(S)$. For each such special vertex v , let $B(v)$ denote a second derived neighborhood of v . Then $\partial B(v)$ meets $S \cup h(S)$ transversally and we may view $(S \cup h(S)) \cap B(v)$ as the join $v * [(S \cup h(S)) \cap \partial B(v)]$. We want to move $h(S)$ into general position with respect to S , using an isotopy that is constant off the balls $B(v)$. Let T denote the image of $h(S)$ after the isotopy. Although T is no longer equal to $h(S)$, we retain careful control of the situation. In particular, we may assume that for each special vertex v we have $T \cap B(v) = v' * (\partial B(v) \cap h(S))$, where v' is the image of the vertex v after the isotopy.

Now S and T are two surfaces homotopic in M such that $S \cap T$ consists of mutually disjoint, simple, closed curves. It follows from Proposition 5.4 of [13] that there exists a surface H and an embedding $H \times I \rightarrow M$ such that $H \times 0 = \tilde{S}' \subset S$, $\overline{(\partial(H \times I) - (H \times 0))} = \tilde{T}' \subset T$, and $\tilde{T}' \cap S = \partial \tilde{T}'$. That is, we have a piece \tilde{S}' of S that is parallel to a piece \tilde{T}' of T . Now we consider the corresponding pieces of S and $h(S)$. Let \tilde{T} be the surface contained in $h(S)$ that is obtained from \tilde{T}' by replacing $\tilde{T}' \cap B(v)$ with $[\tilde{T}' \cap \partial B(v)] * v$ for each special vertex v . Similarly, we obtain \tilde{S} from \tilde{S}' by replacing $\tilde{S}' \cap B(v)$ with $[\tilde{S}' \cap \partial B(v)] * v$ for each special vertex v . There is clearly an isotopy of M , constant on $(S - \tilde{S})$, that carries \tilde{S} to \tilde{T} .

Let G be the surface $(S - \tilde{S}) \cup \tilde{T}$. The above isotopy of M carries S to G . If $h(S - \tilde{S}) = \tilde{T}$ then G is invariant under h . That this is indeed the case we shall now prove by showing that the assumption $h(G) \neq G$ implies the existence of a surface in Σ having lower complexity than S , which is in contradiction to our choice of S .

Let us assume then that $h(G) \neq G$. Let Q denote the graph $\tilde{T} \cap \tilde{S}$. We use two constructions, depending on whether or not Q is invariant under h . In both cases we construct a surface G' in Σ having a lower complexity than that of S . It will be helpful to keep in mind that either $h(\tilde{S}) = \tilde{T}$ or $h(\tilde{S}) \cap \tilde{T} \subset Q$.

Case 1. $h(Q) = Q$. In this case it follows from our assumption $h(G) \neq G$ that $h(\tilde{S}) = \tilde{T}$. We shall proceed to move G into h -general position by an isotopy that is constant outside a regular neighborhood of Q . Take a small invariant regular neighborhood U of Q , which we find convenient to view in the following way. A component of U that contains a special vertex consists of the union of mutually disjoint beams $X_i = D^2 \times I$ together with the ball neighborhoods $B(v)$ which meet the beams along the ends $D^2 \times \partial I$. The remaining components of U are solid tori Y_i . Then $X_i \cap (S \cup h(S))$ consists of two disks that cross along an arc properly embedded in each. Similarly, $Y_i \cap (S \cup h(S))$ consists of two annuli that cross along a boundary parallel, simple, closed curve.

In each beam $X_i = D^2 \times I$ we choose a properly embedded disk C_i satisfying the following properties: (i) $C_i \cap (\partial D^2 \times I) = G \cap (\partial D^2 \times I)$; (ii) $C_i \cap (D^2 \times \partial I) = k_1 \cup k_2$, two arcs with $k_i \cap (S \cup h(S)) = k_i \cap G = \partial k_i$; (iii) $\text{Int}(C_i) \cap (S \cup h(S)) = \emptyset$. In each torus Y_i we choose a properly embedded annulus A_i such that $\partial A_i = G \cap \partial Y_i$ and $\text{Int}(A_i) \cap (S \cup h(S)) = \emptyset$. In the present case it follows that $h(A_i) \cap A_i = \emptyset$ and $h(C_i) \cap C_i = \emptyset$.

Let $E = [(S \cup \tilde{T}) - (\tilde{S} \cup U)] \cup (\bigcup C_i) \cup (\bigcup A_i)$. Then E is a surface whose boundary components lie in the boundaries of those balls $B(v)$ contained in U . We extend E into $B(v)$ to form G' by adjoining $v * (E \cap \partial B(v))$ for each special vertex $v \in U$. That is, we obtain a new surface G' isotopic to G by setting $G' = E \cup \bigcup_{v \in U} (v * (E \cap \partial B(v)))$. This new surface G' may be tangent to $\text{Fix}(h)$ at some of the special vertices. If this happens at v then we can pull G' off $\text{Fix}(h)$ near v by an isotopy constant outside $B(v)$. This can be done in such a way that afterwards G' belongs to Σ and

$$h(G') \cap G' \subset (h(S) \cap S) - (Q - \bigcup \{v | v \text{ a special vertex}\}).$$

Thus we have $c(G') < c(S)$, which is our desired contradiction.

Case 2. $h(Q) \neq Q$. Let P denote $Q \cap h(Q)$, which may be vacuous. Take an invariant regular neighborhood U of P . As in Case 1, we view U as the union of balls $B(v)$, beams $X_i = D^2 \times I$, and solid tori Y_j . Let V be a regular neighborhood of $Q - P$, similarly viewed as consisting of balls $B(v)$, beams X'_i (adjoining the balls), and solid tori Y'_j . We assume the neighborhood V is chosen such that the beams $\{X_i, X'_k, h(X'_j)\}$ are pairwise disjoint and the tori $\{Y'_i, h(Y'_j)\}$ are pairwise disjoint.

In $(\overline{M - (U \cup V)})$ we find a surface \hat{E} close to and parallel to $\overline{\tilde{T} - (U \cup V)}$ such that (i) $\hat{E} \cap (H \times I) = \emptyset$, (ii) $\hat{E} \cap (\partial U \cup \partial V)$ consists of simple closed curves which are parallel to the corresponding curves of $\tilde{T} \cap (\partial U \cup \partial V)$, and (iii) $\hat{E} \cap (S \cup h(S)) = \emptyset$. We will extend the surface \hat{E} into $U \cup V$ to form E .

Consider a beam $X_i = D^2 \times I$ in U (note that X_i is invariant under h only if X_i meets $\text{Fix}(h)$ in a disk). In X_i we choose a disk C_i such that (i) C_i is linear

in each simplex, (ii) $C_i \cap (\partial D^2 \times I) = \hat{E} \cap \partial X_i$, (iii) $C_i \cap (S \cup h(S)) = X_i \cap S \cap h(S)$, and (iv) $C_i \cap (D^2 \times \partial I)$ consists of two arcs in ∂C_i . Observe that if $h(X_j) = X_i$ then $[(S - \tilde{S}) \cap X_i] \cup C_i$ meets $h([(S - \tilde{S}) \cap X_j] \cup C_j)$ transversally along the arc $S \cap h(S) \cap X_i$.

Now consider a beam $X'_i = D^2 \times I$ in V . We choose a disk C'_i satisfying the properties (i) C'_i is linear in each simplex, (ii) $C'_i \cap (h(S) \cup S) = C'_i \cap S = k_i$, an arc in $(S - \tilde{S}) \cap X'_i$ parallel to the arc $X'_i \cap h(S) \cap S$, (iii) $C'_i \cap (\partial D^2 \times I) = \hat{E} \cap (\partial D^2 \times I)$ and (iv) $C'_i \cap (D^2 \times \partial I)$ consists of two arcs in $\partial C'_i$. For later reference, let L_i denote the disk in $(S - \tilde{S}) \cap X'_i$ that is the closure of the component of $S - (\tilde{S} \cup k_i)$ adjacent to \tilde{S} . Observe that $h(C'_i)$ is disjoint from S .

In the solid torus Y_i of U we find an annulus A_i with one boundary component $\hat{E} \cap \partial Y_i$ and the other $Y_i \cap h(S) \cap S$. Moreover, we choose Y_i such that $Y_i \cap (S \cup h(S)) = Y_i \cap h(S) \cap S$, and if $h(Y_j) = Y_i$ then $h(A_j \cup [(S - \tilde{S}) \cap Y_j])$ meets $A_i \cup [(S - \tilde{S}) \cap Y_i]$ transversally along the simple closed curve $Y_i \cap S \cap h(S)$.

Finally, in the solid torus Y'_i of V we take an annulus A'_i parallel to $\tilde{T} \cap Y'_i$ such that (i) $\partial A'_i$ consists of the curve $\hat{E} \cap \partial Y'_i$ and a simple closed curve d_i in $(S - \tilde{S}) \cap Y'_i$ parallel to $Y'_i \cap h(S) \cap S$, and (ii) $A'_i \cap (S \cup h(S)) = d_i$. Let K_i denote the annulus in $(S - \tilde{S})$ bounded by the curves $Y'_i \cap h(S) \cap S$ and d_i . Notice that $h(A'_i)$ is disjoint from S .

We let E^* denote the surface $\hat{E} \cup (\bigcup_i C_i) \cup (\bigcup_i C'_i) \cup (\bigcup_i A_i) \cup (\bigcup_i A'_i)$. For each $v \in U$ we add the join $v * (E^* \cap \partial B(v))$ to E^* to form E . Now define the surface G' , which is isotopic to G , by setting $G' = E \cup (S - (\tilde{S} \cup L \cup K \cup [\bigcup_v (v * (L \cap \partial B(v)))]))$, where L and K denote the union of the disks L_i and the annuli K_i , respectively. As in Case 1, the surface G' may be tangent to $\text{Fix}(h)$ at some of the special vertices v . However we can again pull G' off $\text{Fix}(h)$ by an isotopy constant outside the balls $B(v)$ of the vertices involved to move G' into h -general position. This can be done so that we have $h(G') \cap G' \subset h(S) \cap S - [Q - (P \cup \{v | v \text{ a special vertex}\})]$. Then we have $c(G') < c(S)$ and a contradiction. This completes the proof.

3. Involutions on $M(\phi)$. Before proving Theorem 2 in this section we give some useful lemmas concerning the 3-manifolds $M(\phi) = F \times R^1/\phi$, where F is a closed orientable surface. We show that given any two fiberings of $M(\phi)$ over S^1 there exists an isotopy of $M(\phi)$ that carries one fibering onto the other.

Consider the homeomorphism $\phi: F \rightarrow F$ of the closed surface F . Then ϕ induces an automorphism ϕ_* of $H_1(F; Q)$, the first homology group of F with rational coefficients Q . There is also the homomorphism $I - \phi_*: H_1(F; Q) \rightarrow H_1(F; Q)$. Given a basis for the vector space $H_1(F; Q)$ we can identify $I - \phi_*$ with the corresponding matrix. Then $I - \phi_*$ is an isomorphism if and only if $\det(I - \phi_*) \neq 0$.

The first lemma gives us a convenient way by which we can identify those maps ϕ for which $H_1(M(\phi); Q) \cong Q$.

LEMMA 1. $H_1(M(\phi); Q) \cong Q$ if and only if $\det(I - \phi_*) \neq 0$.

PROOF. If we let $\phi_\#$ denote the automorphism of $\pi_1(F)$ induced by ϕ (well defined up to inner automorphism) then we have the group presentation

$$\pi_1(M(\phi)) \cong \langle \pi_1(F), t : tx t^{-1} = \phi_\#(x), x \in \pi_1(F) \rangle.$$

follows that

$$H_1(M(\phi); Q) \cong (H_1(F; Q) : (I - \phi_*)(x) = 0, x \in H_1(F; Q)) \oplus Q.$$

Therefore, $H_1(M(\phi); Q) \cong Q$ if and only if $I - \phi_*$ is an epimorphism and hence an isomorphism.

LEMMA 2. Suppose that $M(\phi) = F \times R^1 / \phi$ and $H_1(M(\phi); Q) \cong Q$. Let $\psi: G \rightarrow G$ be a homeomorphism of a closed surface G . If there exists a homeomorphism $h: M(\psi) \rightarrow M(\phi)$ then $F \approx G$ and $h([G \times 0])$ is isotopic to $[F \times 0]$ in $M(\phi)$. Moreover, ψ is isotopic to a conjugate of either ϕ or ϕ^{-1} .

PROOF. Since $\pi_1(M(\psi))$ can be given the group presentation $\langle \pi_1(G), s : sxs^{-1} = \psi_\#(x), x \in \pi_1(G) \rangle$, the subgroup $h_\#(\pi_1([G \times 0]))$ is equal to the kernel of the natural homomorphism $\pi_1(M(\psi)) \rightarrow H_1(M(\phi); Q)$. Similarly, $\pi_1([F \times 0])$ is equal to this kernel. This implies that $F \approx G$.

Consider the covering space $P: F \times R^1 \rightarrow M(\phi)$, defined by $P(x, t) = [x, t]$, corresponding to the subgroup $\pi_1([F \times 0])$. Then $h([G \times 0])$ is evenly covered by $P^{-1}(h([G \times 0]))$. There exists a component F' of $P^{-1}(h([G \times 0]))$ disjoint from the compact surface $F \times 0$ in $F \times R^1$. It follows from [3] that there is a submanifold X of $F \times R^1$ such that $X \approx F \times I$ and $\partial X = F' \cup F \times 0$. Thus $P|X$ defines a homotopy from $[F \times 0]$ to $h([G \times 0])$ in $M(\phi)$. By Corollary 5.5 of [13] there exists an isotopy of $M(\phi)$ carrying $[F \times 0]$ to $h([G \times 0])$.

Now consider the isomorphism $h_\#: \pi_1(M(\psi)) \rightarrow \pi_1(M(\phi))$, using the presentations for these two groups given above and in the proof of Lemma 1. As we have already noted, $h_\#(\pi_1(G)) = \pi_1(F)$. It follows that $h_\#(s) = ut^\epsilon$, where $u \in \pi_1(F)$ and $\epsilon = \pm 1$. Hence, we have

$$h_\#(\psi_\#(x)) = h_\#(sxs^{-1}) = ut^\epsilon h_\#(x)t^{-\epsilon}u^{-1} = u\phi_\#^\epsilon(h_\#(x))u^{-1}.$$

Thus $h_\#(\psi_\#(h_\#^{-1}(y))) = u\phi_\#^\epsilon(y)u^{-1}$, for $y \in \pi_1(F)$. The automorphisms $h_\#\psi_\#h_\#^{-1}$ and $\phi_\#^\epsilon$ differ only by an inner automorphism of $\pi_1(F)$. If we let $g: G \rightarrow F$ be a homeomorphism inducing the isomorphism $h_\#|_{\pi_1(G)}$, then it follows from a theorem of Baer [1] that $g\psi g^{-1}$ is isotopic to ϕ^ϵ .

PROOF OF THEOREM 2. It follows from Theorem 1 and Lemma 2 that there exists a fiber F of M such that either $h(F) = F$ or $h(F) \cap F = \emptyset$. Suppose that

$h(F) = F$. Split M along F to obtain $F \times I$. It follows from Theorem 3 that there exists a map β of F such that h determines an involution g on $F \times I$ defined by $g(x, t) = (\beta(x), \lambda(t))$, where $\lambda(t) = t$ or $1 - t$. Let θ be the map repairing the cut along F , where θ is defined with respect to the given product structure. If $h(F) \cap F = \emptyset$, then we split M along $F \cup h(F)$ to obtain two copies of $F \times I$, which we parametrize as $F \times [0, \frac{1}{2}]$ and $F \times [\frac{1}{2}, 1]$. We may assume that the second product structure is chosen in such a way that h maps (x, t) to $(\beta(x), t + \frac{1}{2})$ for $0 \leq t \leq \frac{1}{2}$. Then we may view M as $F \times I/\theta$ where $h([x, t]) = [\beta(x), t + \frac{1}{2}]$, for $0 \leq t \leq \frac{1}{2}$. Since $h^2 = \text{identity}$, we must have $\beta^2 = \theta$.

In either case, it follows from Lemma 2 that θ is isotopic to a conjugate of ϕ^e , say $\gamma\phi^e\gamma^{-1}$. Consider the homeomorphism $\psi = \gamma^{-1}\theta^e\gamma$ of F , which is clearly isotopic to ϕ . Then h is equivalent to the involution h' defined on $F \times R^1/\psi$ by $h'([x, t]) = [\alpha(x), \lambda(t)]$, where $\alpha = \gamma^{-1}\beta\gamma$. This completes the proof of Theorem 2.

EXAMPLE 1. We give an example to show that given a PL involution of $M(\phi) = F \times R^1/\phi$ there need not always exist an equivariant fibering with fiber F . However, it is probably true that one can always find an equivariant fibering over S^1 (with fiber possibly distinct from F). Let F be a closed orientable surface of genus $g \geq 2$. Consider a two-sheeted covering space $p: \tilde{F} \rightarrow F$ with nontrivial covering transformation ψ . Then $F \times S^1$ is homeomorphic to $M(\psi) = \tilde{F} \times R^1/\psi$ [10]. Let h be the involution of $M(\psi)$ defined by $h([x, t]) = [x, 1 - t]$. Then $\text{Fix}(h)$ is homeomorphic to \tilde{F} . However, if we view $M(\psi)$ as $F \times S^1$, it is impossible to find a fiber F such that $h(F) = F$ or $h(F) \cap F = \emptyset$ in view of Theorem 3.

4. Realizing homotopy classes of finite order by periodic maps. Let $\phi: F \rightarrow F$ be a homeomorphism of the closed orientable surface F and consider the 3-manifold $M(\phi) = F \times R^1/\phi$. If $H_1(M(\phi); Q) \cong Q$ and $\chi(F) < 0$ we prove (Theorem 4) that all cyclic subgroups of prime order in $\text{Out}(\pi_1(M(\phi)))$ are realized by periodic homeomorphisms of $M(\phi)$. The example of Raymond and Scott [9] for which this fails is the torus bundle $M(\psi)$ determined by the map $\psi(z_1, z_2) = (z_1, z_1^2 z_2)$, where we are viewing $S^1 \times S^1$ as $\{(z_1, z_2) \in C^2 : |z_1| = |z_2| = 1\}$. They show that there exists a homeomorphism h of $M(\psi)$ such that $h^2 \simeq 1$ but h is not homotopic to any involution. It would be interesting to characterize all such exceptions as this. The next lemma and theorem give an indication as to why this torus bundle example works.

LEMMA 3. *Let ϕ and g be homeomorphisms of the closed orientable surface F and let g fix a basepoint x_0 of F . Suppose that h and h' are homeomorphisms of $M(\phi)$ such that $h([x, 0]) = h'([x, 0]) = [g(x), 0]$ and the composition $h \circ h'$ does not interchange the sides of the surface $[F \times 0]$. Assume that $\phi g = g \phi$ if h does not interchange the sides of $[F \times 0]$ and $\phi g = g \phi^{-1}$ otherwise. If $\chi(F) < 0$ then h is isotopic to h' .*

PROOF. If we split $M(\phi)$ along $[F \times 0]$ to obtain $F \times I$ then h and h' define maps \hat{h} and \hat{h}' on $F \times I$ such that $(x, 0) \mapsto (\phi^\gamma g(x), \gamma)$ and $(x, 1) \mapsto (\phi^\gamma g(x), 1 - \gamma)$, where $\gamma = 0$ or 1 . Let H denote the composition

$$F \times I \xrightarrow{\hat{h}} F \times I \xrightarrow{\text{proj}} F \xrightarrow{(\phi^\gamma g)^{-1}} F.$$

Observe that $H(x, 0) = H(x, 1) = x$ for $x \in F$. Then $H|_{x_0 \times I}$, the trace of x_0 under the homotopy H , represents an element in the center of $\pi_1(F, x_0)$ [11]. But this group is centerless since $\chi(F) < 0$. Hence $H|_{x_0 \times I}$ represents the trivial element in $\pi_1(F, x_0)$ and thus $h|_{[x_0 \times I]}$ is homotopic to $[x_0 \times I]$ by a homotopy fixing the basepoint x_0 . Of course, the same is true for $h'|_{[x_0 \times I]}$. It follows that h and h' induce the same automorphism of $\pi_1(M(\phi), x_0)$. Therefore, by Waldhausen [13], the map h is isotopic to h' .

THEOREM 5. Let $M(\phi) = F \times R^1/\phi$, where F is a closed orientable surface of negative Euler characteristic and ϕ is any homeomorphism of F . Suppose that h is a map of M such that $h([F \times 0])$ is homotopic to $[F \times 0]$ in $M(\phi)$ and $h^p \simeq 1$ for some prime p . Then there exists a homeomorphism h' of $M(\phi)$ such that h' is homotopic to h and $h'^p = 1$.

PROOF OF THEOREM 5. If $h \simeq 1$ we can set $h' = 1$. So let us assume that $h \not\simeq 1$. According to Theorem 6.1 of [13], the map h is homotopic to a homeomorphism and so we may as well also assume that h is already a PL homeomorphism. We make some adjustments on h and ϕ . Choose a basepoint $x_0 \in F = [F \times 0]$. We may assume that ϕ is chosen such that $\phi(x_0) = x_0$. Since $h([F \times 0])$ is homotopic to $[F \times 0]$, we may deform h by a homotopy such that afterwards we have $h([F \times i/p]) = [F \times i/p]$ and $h([x_0, i/p]) = [x_0, i/p]$ for $i = 0, 1, \dots, p-1$. Let $H: M(\phi) \times I \rightarrow M(\phi)$ denote an isotopy from 1 to h^p . (It follows from [13] that homotopic homeomorphisms of $M(\phi)$ are isotopic.) To keep track of this isotopy we consider the fibering $f: M(\phi) \rightarrow S^1$ defined by $f([x, t]) = [t]$, where $S^1 = R/Z$. Let $\Sigma = \{x_0\} \times S^1$ and define the map $H|_\Sigma: \Sigma \rightarrow M(\phi)$ by $H|_\Sigma((x_0, [t])) = H(x_0, t)$ for $0 \leq t \leq 1$. Let n denote the degree of $f \circ H|_\Sigma$. We want to change h by an isotopy in order to reduce this degree to 0 . We accomplish this by composing h with the homeomorphism $\lambda_s: M(\phi) \rightarrow M(\phi)$ defined by $\lambda_s([x, t]) = [x, t + s]$, for an appropriate choice of s . Clearly λ_s is isotopic to the identity.

Case 1. $n \not\equiv 0 \pmod{p}$. There is a unique integer a such that $0 < a < p$ and $an \equiv -1 \pmod{p}$. If we define $h_1 = h \circ \lambda_{n/p}^{-1}$ then there exists a homotopy H' from 1 to h_1^p such that the degree of $f \circ H'|_\Sigma$ is 0 . Since $h_1([x, t]) = h([x, t - n/p])$, we have $h_1^a([F \times i/p]) = [F \times (i/p - na/p)] = [F \times (i + 1)/p]$. If we can find a periodic map h' homotopic to h_1^a then $(h')^{-n}$ is a periodic map

homotopic to the original h . To simplify our notation, there is no harm in assuming that our original map h already has the form of h_1^a . Thus let us assume that (i) $h([F \times i/p]) = [F \times (i+1)/p]$, (ii) $h([x_0, i/p]) = [x_0, (i+1)/p]$ and (iii) there is a homotopy $H : 1 \simeq h^p$ such that $f \circ H|_\Sigma$ has degree 0.

We can redefine the local product structure of $M(\phi)$ on $[F \times [1/p, 1]]$ such that, for $0 \leq t \leq 1/p$, h is defined by $h([x, t]) = [g(x), t + 1/p]$, where $g = h|_{[F \times 0]} : [F \times 0] \rightarrow [F \times 1/p]$. This reparametrization changes the map ϕ , but we continue to denote it by ϕ . Notice that we still have $\phi(x_0) = x_0$ since $h([x_0, 1/p]) = [x_0, 2/p]$.

Consider the lifting map \tilde{h} on $F \times R^1$ with $\tilde{h}(x_0, 0) = (x_0, 1/p)$ which covers h with respect to the covering space $P : F \times R^1 \rightarrow M(\phi)$ defined by $P(x, t) = [x, t]$. Then \tilde{h}^p covers h^p and the homotopy $H : 1 \simeq h^p$ can be lifted to a homotopy \tilde{H} ending with h^p . Since the degree of $f \circ H|_\Sigma$ is 0 it follows that $\tilde{H}_0(x_0, 0) = (x_0, 1)$ and thus \tilde{H}_0 is the covering transformation $(x, t) \mapsto (\phi(x), t + 1)$. Observe that $\tilde{h} \simeq g \times 1$ and $\tilde{h}^p \simeq \phi \times 1$. Hence g^p is homotopic to ϕ .

It is convenient to work now with $M(g^p)$, a homeomorphic copy of $M(\phi)$. Let $G : F \times I \rightarrow F$ be an isotopy from $G_0 = 1$ to $G_1 = g^p \phi^{-1}$. We define the homeomorphism $f : M(\phi) \rightarrow M(g^p)$ by

$$f([x, t]) = \begin{cases} [x, t], & \text{for } 0 \leq t \leq 1 - 1/\epsilon, \\ [G(x, \epsilon t + (1 - \epsilon)), t], & \text{for } 1 - 1/\epsilon \leq t \leq 1, \end{cases}$$

where ϵ is chosen to satisfy $\epsilon > p$.

We define two new maps h' and h'' on $M(g^p)$ by setting $h' = f \circ h \circ f^{-1}$ and $h''([x, t]) = [g(x), t + 1/p]$. Observe that $h'([x, 0]) = [g(x), 1/p]$. It thus follows from Lemma 3 that h' is isotopic to h'' . Therefore h is isotopic to the map $f^{-1} \circ h'' \circ f$, which is easily seen to be periodic.

Case 2. $n \equiv 0 \pmod{p}$. There exists an integer k such that $n = pk$. If we define $h_1 = h \circ \lambda_k^{-1}$, then there exists a homotopy H' from 1 to h_1^p such that $f \circ H'|_\Sigma$ has degree 0. We can lift h_1 to a homeomorphism \tilde{h}_1 on $F \times R^1$ such that $\tilde{h}_1(x_0, 0) = (x_0, 0)$. The homotopy H' lifts to a homotopy from 1 to \tilde{h}_1^p . If $g_1 = h_1|_{[F \times 0]}$ then $\tilde{h}_1 \simeq g_1 \times 1$ and hence $g_1^p \simeq 1$. It follows from [8] and [14] that there exists a map g of F such that g is isotopic to g_1 and $g^p = 1$. Let $L : F \times I \rightarrow F$ be an isotopy from g_1 to g .

Subcase (a). h_1 does not interchange the two sides of $[F \times 0]$. We can isotope h_1 , keeping it fixed on $[F \times [3/4, 1]]$, such that afterwards $h_1([x, t]) = [g_1(x), t]$ for $0 \leq t \leq 1/2$. Now we further deform h_1 to a map h_2 by an isotopy constant on $[F \times [1/2, 1]]$ such that $h_2([x, 0]) = [g(x), 0]$. For this, consider $K : M(\phi) \times I \rightarrow M(\phi)$ defined by

$$K([x, t], s) = \begin{cases} [g_1(x), t], & 0 \leq t \leq s/4, \\ [L(x, 4t - s), t], & s/4 \leq t \leq 1/4, \\ [L(x, 2 - s - 4t), t], & 1/4 \leq t \leq (2 - s)/4, \\ [g_1(x), t], & (2 - s)/4 \leq t \leq 1/2, \\ h_1([x, t]), & 1/2 \leq t \leq 1. \end{cases}$$

Then $K_1 = h_1$ and $K_0([x, 1/4]) = [g(x), 1/4]$. If we define $h_2 = \lambda_{1/4}^{-1} \circ K_0 \circ \lambda_{1/4}$ we have $h_2 \simeq h_1$ and $h_2([x, 0]) = [g(x), 0]$.

Now split $M(\phi)$ along $[F \times 0]$ to obtain $F \times I$ on which h_2 defines a map $\hat{h}_2 : F \times I \rightarrow F \times I$ such that $\hat{h}_2(x, 0) = (g(x), 0)$ and $\hat{h}_2(x, 1) = (\phi g \phi^{-1}(x), 1)$. In fact, \hat{h}_2 is a homotopy between g and $\phi g \phi^{-1}$. According to Theorem 3 in [2], if we make the proper choice of g in the beginning, there exists a homeomorphism ψ isotopic to ϕ such that $g\psi g^{-1} = \psi$. Let $G : F \times I \rightarrow F$ be an isotopy from 1 to $\psi \phi^{-1}$. Define the homeomorphism $f : M(\phi) \rightarrow M(\psi)$ by

$$f([x, t]) = \begin{cases} [x, t], & 0 \leq t \leq 1/2, \\ [G(x, 2t - 1), t], & 1/2 \leq t \leq 1. \end{cases}$$

Define homeomorphisms on $M(\psi)$ by $h' = f \circ h_2 \circ f^{-1}$ and $h''([x, t]) = [g(x), t]$. Since $h'([x, 0]) = [g(x), 0]$, it follows from Lemma 3 that h' is isotopic to h'' . Moreover, $(h'')^p = 1$ and h is isotopic to $f^{-1} \circ h'' \circ f$.

Subcase (b). h_1 interchanges the sides of $[F \times 0]$. Since $h_1^p \simeq 1$, the sides of $[F \times 0]$ are not interchanged by h_1^p [13]. Thus p must be equal to 2. We can isotope h_1 such that $h_1|_{[F \times [0, 1/4]]} : [x, t] \mapsto [g_1(x), -t]$. We shall deform h_1 further using the function $K : M(\phi) \times I \rightarrow M(\phi)$ defined by

$$K([x, t], s) = \begin{cases} [g_1(x), -t], & 0 \leq t \leq s/8, \\ [L(x, 8t - s), -t], & s/8 \leq t \leq 1/8, \\ [L(x, 2 - s - 8t), -t], & 1/8 \leq t \leq (2 - s)/8, \\ [g_1(x), -t], & (2 - s)/8 \leq t \leq 1/4, \\ h_1([x, t]), & 1/4 \leq t \leq 1. \end{cases}$$

We define $h_2 = \lambda_{1/8} \circ K_0 \circ \lambda_{1/8}$. Observe that h_2 is isotopic to h_1 and $h_2([x, 0]) = [g(x), 0]$. Now split $M(\phi)$ along $[F \times 0]$ to obtain $F \times I$. There is the map \hat{h}_2 defined on $F \times I$ such by h_2 such that $\hat{h}_2(x, 0) = (\phi g(x), 1)$ and $\hat{h}_2(x, 1) = (g\phi^{-1}(x), 0)$. Thus \hat{h}_2 is a homotopy between ϕg and $g\phi^{-1}$. One can show by using nearly the same argument as used for the proof of Theorem 3 in [2] that

g could have been chosen such that there exists a homeomorphism ψ of F isotopic to ϕ for which $\psi g = g\psi^{-1}$. Now complete the argument just as we did in subcase (a) by defining maps h' and h'' on $M(\psi)$, except this time the periodic map h'' will be defined by $h''([x, t]) = [g(x), -t]$.

REMARK. Theorem 4 follows immediately now in view of Lemma 2. Consider $M(\phi) = F \times R^1/\phi$, where $\chi(F) < 0$. Given a map h of $M(\phi)$ such that $h^p \simeq 1$, all one needs in order to apply Theorem 5 is that $h([F \times 0])$ be homotopic to $[F \times 0]$ in $M(\phi)$. Lemma 2 gives this to us whenever $H_1(M(\phi); \mathbb{Q}) \cong \mathbb{Q}$.

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